2011 Marywood Mathematics Contest
Level II
Sponsored by
SEMI-GROUP
The Student Mathematics Club of
Marywood University
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Directions:

1. This exam consists of 40 questions on 6 pages. Please check to make sure that you have all the pages.

2. No calculator or any other electronic device is allowed on this exam.

3. Allot your time accordingly. This is a 60-minute test. Do not spend too much time on any one problem. If a question seems to be too difficult, make your best possible guess. Your score will be the number of correct responses.

4. On the scantron form provided for you, darken in the space corresponding to the correct answer. Please mark all answers carefully and erase completely when changing an answer. Mark only one answer for each question. Only those answers on the answer sheet will be counted.

5. There is a sheet of blank paper on the last page which you can tear off and use as scratch paper. You may also use the back of the pages.

6. NOTE: In order to ensure uniformity, proctors are NOT allowed to answer any questions pertaining to specific problem content.

Please do NOT open the test until you are told to do so.
1. \((-2)^{-2} + (-3)^{-3} =\)
   A. \(-\frac{23}{108}\)  B. \(\frac{23}{108}\)  C. \(-\frac{31}{108}\)  D. \(\frac{31}{108}\)  E. None of these.

2. For two consecutive days, Thomas ate 20% of the jellybeans that were in his jar at the beginning of that day. There were 32 remaining at the end of the second day. How many jellybeans were in the jar originally?
   A. 40  B. 50  C. 55  D. 60  E. 75

3. Given that \(\log_{10} 2011 \approx 3.303\), how many digits does \(2011^{10}\) have (in base 10)?
   A. 30  B. 33  C. 34  D. 40  E. None of these.

4. If a right triangle has one angle equal to 30° and the length of its hypotenuse is 8, what is the length of the altitude that is perpendicular to the hypotenuse?
   \[\triangle \text{ with } 30° \quad \text{and hypotenuse } 8\]
   A. 2  B. 3  C. \(2\sqrt{3}\)  D. \(3\sqrt{3}\)  E. None of these.

5. Solve the equation \(\log_{10} x + \log_{10}(x - 2) = \log_{10} 3\), to get \(x = \)
   A. 3.  B. 1.  C. \(-1.\)  D. 3 or \(-1.\)  E. 3 or 1.

6. \(\sqrt[3]{x \sqrt{x \sqrt{x}}} =\)
   A. \(x^{13/27}\)  B. \(x^{1/27}\)  C. \(x^{1/9}\)  D. \(x^{13/9}\)  E. None of these.

7. If \(\cos \theta = 0.6\) and \(0 < \theta < \pi/2\), what is \(\tan \theta\)?
   A. 0.8  B. 0.75  C. \(2/3\)  D. \(4/3\)  E. None of these.

8. Find the sum of all integers from 1 to 2011: \(1 + 2 + 3 + \cdots + 2010 + 2011 =\)
   A. \(2011 \times 1006\)  B. \(2011 \times 1005\)  C. \(2012 \times 1006\)  D. \(2012 \times 1005\)  E. None of these.
9. If the two diagonals of a rectangle make an angle of 40°, what is the ratio between two adjacent sides of the rectangle?

A. \( \sin 20° : \sin 70° \)  
B. \( \sin 40° : \sin 140° \)  
C. \( \cos 20° : \cos 70° \)  
D. \( \cos 40° : \cos 140° \)  
E. None of these.

10. The equation \( x^2 - 2011x + 2010 = 0 \) has

A. no real solution.  
B. 1 real solution.  
C. 2 real solutions.  
D. 3 real solutions.  
E. None of these.

11. Given two functions \( f(x) \) and \( g(x) \), the composite function \( (f \circ g)(x) \) is defined as \( (f \circ g)(x) = f(g(x)) \). If \( f(x) = x^2 + 5 \) and \( g(x) = \frac{1}{x+1} \), what is \( (f \circ g)(x) \)?

A. \( \frac{1}{(x+1)^2 + 5} \)  
B. \( \frac{x^2 + 5}{x+1} \)  
C. \( \frac{1}{x^2 + 5 + 1} \)  
D. \( \frac{1}{(x+1)^2 + 5} \)  
E. \( \frac{x+1}{x^2 + 5} \)

12. There are 5 white beans, 9 black beans, and 7 red beans in a bag. If one randomly takes out a certain number of beans without looking, what is the minimum number of beans needed to guarantee that all three colors are represented among the beans taken out?

A. 13  
B. 15  
C. 16  
D. 17  
E. 18

13. If a fair coin is tossed 3 times, what is the probability that one will observe at least two heads consecutively OR at least two tails consecutively?

A. 3/8  
B. 1/2  
C. 5/8  
D. 3/4  
E. None of these.

14. In the figure below, the smaller circle passes through the center of the larger circle and its diameter is equal to the radius of the larger circle. The center of the smaller circle \( C \) is on the diameter \( AB \) of the larger circle. \( AE \) is tangent to the smaller circle at \( D \). What is \( \cos \angle BAE \)?

[Diagram of circles and points A, B, O, C, D, E]

A. \( 2\sqrt{2}/3 \)  
B. 1/3  
C. 8/9  
D. \( 1/(2\sqrt{2}) \)  
E. None of these.
15. If \( \cos x = \frac{1}{4} \), and \( 0 < x < \pi/2 \), what is \( \sin(2x) \)?

A. \( \frac{\sqrt{15}}{16} \)  
B. \( \frac{\sqrt{15}}{8} \)  
C. \( -\frac{\sqrt{15}}{16} \)  
D. \( -\frac{\sqrt{15}}{8} \)  
E. \( \frac{\sqrt{15}}{4} \)

16. If a circle centered in the first quadrant passes through the points \((0,1)\) and \((0,5)\) on the \( y \)-axis, and it is tangent to the \( x \)-axis, where is the center?

A. \((3,3)\)  
B. \((3,\sqrt{5})\)  
C. \((\sqrt{5},3)\)  
D. \((\sqrt{8},3)\)  
E. None of these.

17. The area enclosed by the graph of \( y = 3|x| - 5 \) and the \( x \)-axis is

A. \( \frac{25}{6} \)  
B. \( \frac{25}{9} \)  
C. \( \frac{25}{18} \)  
D. \( \frac{25}{12} \)  
E. \( \frac{25}{3} \)

18. Find the intersection point \((x_0, y_0)\) of the lines \( y = -2x + 1 \) and \( x + 3y = -7 \), then determine the value of \( x_0 + y_0 \).

A. \(-5\)  
B. \(-1\)  
C. \(5\)  
D. \(1\)  
E. None of these.

19. A circle of radius 1 is centered at \((3,4)\). What is the distance from the origin \((0,0)\) to the point on the circle closest to the origin?

A. 3  
B. 4  
C. 5  
D. 6  
E. 7

20. For what value of \( k > 0 \) will the circle \( x^2 + y^2 = 3k \) and the line \( y = x + k \) be tangent to each other?

A. 8  
B. 6  
C. 4  
D. 2  
E. None of these.
21. If a circle and a regular hexagon (six sided shape with equal side length) have equal area, and the circle radius is 3, what is the side length of the hexagon?

A. $2\sqrt{3}\pi$ B. $\frac{27\sqrt{3}}{2}\pi$ C. $\sqrt{2}\pi \cdot \sqrt{3}$ D. $\sqrt{\frac{27\pi}{2}} \cdot \sqrt{3}$ E. None of these.

22. Find the remainder of $x^{2011} - x^{2010} + 1$ divided by $x - 2$.

A. $2^{2010}$ B. $2^{2010} + 1$ C. $2^{2011}$ D. $2^{2011} + 1$ E. None of these.

23. There are two math teachers and 3 physics teachers in a meeting sitting at a round table. If the two math teachers cannot sit next to each other, how many distinct ways can the seats be arranged, NOT counting rotations of the same seating.

A. 12 B. 24 C. 96 D. 120 E. None of these.

24. If a parabola representing the quadratic equation $y = x^2 + ax + b$ has $y$-intercept equal to 12, and one of the $x$-intercepts is 3, what is the value of $a + b$?

A. 5 B. $-7$ C. 4 D. 7 E. None of these.

25. Find the radius of the circle inscribed in the triangle with sides 5, 12, and 13.

A. 1 B. 1.5 C. 2 D. 2.5 E. 3

26. Suppose that a function $f(x)$ satisfies $3f(x) + 2f(1 - x) = 2x + 9$ for every real number $x$. What is the value of $f(1)$?

A. 0 B. 1 C. 2 D. 3 E. 4

27. Along the edges of a $3 \times 3$ square, there are 12 lattice points arranged as in the figure below. How many triangles can be formed with these lattice points?

A. 48 B. 64 C. 204 D. 220 E. 256
28. Let \( r \) and \( s \) be the two roots of the equation \( x^2 + 5x + 2 = 0 \). Find \( r^3 + s^3 \).

A. \(-95\)  
B. \(95\)  
C. \(115\)  
D. \(-115\)  
E. None of these.

29. Given \( 3 = \sqrt{a} + \frac{1}{\sqrt{a}} \) where \( a > 1 \), what is \( a - \frac{1}{a} \)?

A. \(5\)  
B. \(6\)  
C. \(3\sqrt{5}\)  
D. \(7\)  
E. \(5\sqrt{2}\)

30. Let \( i \) be \( \sqrt{-1} \), what is \( i + i^2 + i^3 + i^4 + \cdots + i^{2010} + i^{2011} \)?

A. \(1 + i\)  
B. \(1 - i\)  
C. \(-1 + i\)  
D. \(-1 - i\)  
E. None of these.

31. Consider the periodic continued fraction

\[ x = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]

The value of \( x \) is equal to

A. \(-1 + \sqrt{3}\)  
B. \(-1 - \sqrt{3}\)  
C. \(1 + \sqrt{3}\)  
D. \(1 - \sqrt{3}\)  
E. None of these.

32. The stairs leading up to the main entrance of the Marywood University Liberal Arts Center (the building you are in right now) have 9 steps. If one is only allowed to go up 1 step or 2 steps at a time, how many different ways are there to go up the stairs?

A. \(34\)  
B. \(54\)  
C. \(89\)  
D. \(35\)  
E. \(55\)

33. Let \( z = \frac{1}{2} + \frac{\sqrt{3}}{2}i \), where \( i = \sqrt{-1} \). Find \( z^{2011} \).

A. \(\frac{1}{2} + \frac{\sqrt{3}}{2}i\)  
B. \(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\)  
C. \(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\)  
D. \(\frac{1}{2} - \frac{\sqrt{3}}{2}i\)  
E. \(-1\)

34. Let \( x \) be the repeating decimal number \( 3.\overline{19} = 3.191919\cdots \), which can also be represented as a reduced fraction \( \frac{a}{b} \) where \( a \) and \( b \) are relatively prime positive integers. What is \( a + b \)?

A. \(418\)  
B. \(415\)  
C. \(319\)  
D. \(316\)  
E. None of these.
35. From a group of two math teachers and three physics teachers, a committee of three is to be formed. If the selection process is completely random, what is the probability that exactly 2 physics teachers and 1 math teacher are selected to be on the committee?

A. 0.3  B. 0.4  C. 0.5  D. 0.6  E. 0.7

36. Let $u, v, w, x, y$, and $z$ be the degree measures of the six angles of a hexagon. Suppose that $u < v < w < x < y < z$ and $u, v, w, x, y, z$ form an arithmetic sequence. Find the value of $w + x$.

A. 60  B. 72  C. 90  D. 108  E. 240

37. How many pairs of positive integers $(x, y)$ with $y < x \leq 100$ are there such that both $\frac{x}{y}$ and $\frac{x + 1}{y + 1}$ are integers?

A. 86  B. 85  C. 84  D. 83  E. 82

38. If $x, y, z$ satisfy

$$\frac{x}{y - 6} = \frac{y}{z - 8} = \frac{z}{x - 10} = 3.$$

What is the value of $x + y + z$?

A. 24  B. 30  C. 32  D. 36  E. 40

39. How many real numbers $x$ satisfy the following inequality?

$$|x^4 - 4x^2 + 3| \geq |x^4 - 4x^2 + 5|$$

A. Infinitely many.  B. 0  C. 1  D. 2  E. 4

40. How many three-digit integers (numbers between 100 and 999) are there such that the three digits are in strictly increasing order from left to right?

Example: 137 has the digits in strictly increasing order, but 215 or 115 does not have this property.

A. 28  B. 85  C. 56  D. 84  E. None of these.
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